the transverse cross section, the torsional moment takes its limiting value

$$M_n = \frac{1}{4}\tau_T \oint_L \left(\delta^2 - \frac{\delta^3}{3\rho}\right) dL \tag{2.2}$$

The relation connecting the torsional moment M with the twist θ (M_y and θ_y are limiting values of the corresponding quantities, and K is the torsional rigidity within the elastic limits)

$$m = \frac{M - M_y}{M_n - M_y}, \quad \varphi = K\frac{\theta - \theta_y}{M_n - M_y}$$

is shown in Fig.2. The dashed lines show the values of $m(\varphi)$ for the rectangular cross sections, with numbers accompanying the lines describing the ratios of the sides. The half-moons correspond to the Weber profile shown in the same figure. The small circles show the results for the corresponding cross sections with a ratio of the internal to external radii equal to 0.9, and the dark circles refer to the values of m for rods of circular transverse cross section.

The computations show that the curves determining the relationship $m(\varphi)$ are contained within the zone shown in Fig.2 with thick lines. When the twist θ is increased, the plastic zones in which the assumptions made hold exactly, also increase and the magnitude of the torsional moment tends to its exact value (2.2).

We note that the largest error in determining the torsional moment using the formulas given occurs at the yield point. Comparing the values of M_y with the known exact solutions we find that the maximum error is small in the case of simple rods. For prismatic rods of elliptical and rectangular cross section the error does not exceed 3% and 5%, respectively, and for the Weber profile (Fig.2) 1.5%. We note that the torsional moment of a rod of rectangular cross section is determined in /4/ using a more complicated method, yet achieving the same accuracy as in the present paper for a ratio of the sides equal to 0.2 and 0.4. In /5/ the results for a square transverse cross section fall below the limit curve and cannot therefore be regarded as possible.

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EXTENSION OF THE VARIATIONAL FORMULATION OF THE PROBLEM FOR A RIGID-PLASTIC MEDIUM TO VELOCITY FIELDS WITH SLIP-TYPE DISCONTINUITIES*

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Sets of velocity fields containing slip-type discontinuities at the boundary of the rigid-plastic medium, as well as within it, and the functionals defined on these sets, are described. It is shown that the exact lower bounds of the variational problems for these functionals are equal to the coefficient of the critical load. The minimax problem with saddle point constructed here is regarded as an extension of the classical minimax problem of the theory of critical loads.

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It is well-known that the classical variational formulation of the problem of determining the velocity field corresponding to the critical equilibrium state of the rigid-plastic medium is ill-posed in the following sense. In a number of problems there are no smooth velocity fields on which the critical load can be realized (see e.g. /1, 2/). This creates difficulties in computing, since minimizing sequences of the smooth velocity fields must then be used at the limit at which the classical functional determining the upper bound for the critical load coefficient is not defined. A correct formulation of the corresponding variational proble can be obtained directly from the abstract extension scheme formulated in /3/. The results of subsequent investigations of the problem were given in /3/ (for more detail see /1/). As was said in /1/, the variational extensions obtained in these papers cannot be regarded as final and effective, since it is desirable to express them not in terms of the measures, but in terms of the vector function of a point. A development of the classical variational problem which can yield a number of useful practical corollaries is given below.

1. Basic notation and formulation of the problem.

We denote by Ω a region of Euclidean space R^n $(n = 2, 3)$. We assume that the region is bounded and has a boundary Γ which satisfies the Lipshits condition. We denote the velocity field by $u = (u_i)$ and the strain rate tensor by $\varepsilon(u) = (\varepsilon_{ij}(u))$. If a Cartesian coordinate system is chosen, then $2\varepsilon_{ij}(u) = u_{i,j} + u_{j,i}$. We denote the stress tensor by $\sigma = (\sigma_{ij})$ and its first invariant and deviator by σ_{kk} and σ^D. We define the velocity field spaces which will be used below, as follows:

$$D^2(\Omega) = \left\{u:\ \|u\| = \int_\Omega (|u| + |\varepsilon^D(u)|)\,dx + \left(\int_\Omega \operatorname{div}^2 u\,dx\right)^{1/2} < +\infty\right\} \tag{1.1}$$

$$|u|^2 = u_i u_i, \quad |\varepsilon|^2 = \varepsilon_{ij}\varepsilon_{ij}, \ \operatorname{div} u = u_{i,i}$$

The differential operator $\varepsilon_{ij}(u)$ is regarded as a distribution, i.e. for any continuous differentiable function φ with a compact support in Ω,

$$\int_\Omega (u_i\varphi_{,j} + u_j\varphi_{,i})\,dx = -2\int_\Omega \varepsilon_{ij}(u)\varphi\,dx$$

The functional space $D^2(\Omega)$ has the following properties:

a) smooth functions are dense in the space $D^2(\Omega)$ on its norm;

b) the space $D^2(\Omega)$ imbeds completely and continuously into the spaces of summable vector functions $L^r(\Omega)^n$ for $r \in [1, n/(n-1))$, and continuously in $L^{n/(n-1)}(\Omega)^n$;

c) vector functions belonging to $D^2(\Omega)$ have summable traces on Γ, or more accurately, the space $D^2(\Omega)$ imbeds continuously in $L^1(\Gamma)^n$.

We shall study the stress fields defined by symmetric second rank tensors in the following class:

$$\Sigma = \{\tau:\ \tau = (\tau_{ij}),\ \tau_{ij} = \tau_{ji},\ \tau_{kk} \in L^2(\Omega) \\ \tau_{ij}^D \in L^\infty(\Omega),\quad i, j = 1, \ldots, n\} \tag{1.2}$$

We will denote by K the set of all $\tau \in \Sigma$ satisfying the Mises condition (k_* is a given quantity)

$$K = \{\tau \in \Sigma: |\tau^D(x)| \leqslant \sqrt{2}\,k_* \ \text{ for nearly all } \ x \in \Omega\}$$

We assume that the field of distributed loads $f = (f_i)$ is specified in Ω, a field of surface loads $F = (F_i)$ is defined on a part of the surface γ, and

$$f_i \in L^n(\Omega), \quad F_i \in L^\infty(\gamma), \quad i = 1, \ldots, n \tag{1.3}$$

Consider the set of admissible velocity fields

$$V = \{v \in D^2(\Omega): \operatorname{div} v = 0 \text{ in } \Omega,\ v = 0 \text{ on } \Gamma \setminus \gamma, \qquad \int_\Omega f_i v_i\,dx + \int_\gamma F_i v_i\,d\Gamma = 1\}$$

We know (see e.g. /1/) that the critical load coefficient λ_* is found for the rigid-plastic medium from the solution of the following minimax problem:

$$\lambda_* = \inf_V \sup_K \int_\Omega \varepsilon_{ij}(v)\tau_{ij}\,dx = \sup_K \inf_V \int_\Omega \varepsilon_{ij}(v)\tau_{ij}\,dx \tag{1.4}$$

Introducing the notation

$$J(v) = \sup_K \int_\Omega \varepsilon_{ij}(v)\tau_{ij}\,dx = \sqrt{2}\,k_* \int_\Omega |\varepsilon(v)|\,dx$$

$$R(\tau) = \inf_V \int_\Omega \varepsilon_{ij}(v)\tau_{ij}\,dx$$

we find that the functionals $J(v)$ and $R(\tau)$ yield, respectively, the upper and lower bound for the critical load coefficient λ_* and

$$\lambda_* = \inf_V J(v) = \max_K R(\tau) = R(\tau_*)$$

Replacing sup by max means, as usual, that an element $\tau_* \in K$ exists on which the exact upper bound of the functional R is attained. For the functional J, replacing inf by min is, generally speaking, incorrect, since the set V need not contain the velocity field realizing its exact lower bound. This is due to the fact that the class V does not contain velocity fields describing slip-type discontinuities. Moreover, the functional J itself is not defined on such fields. All this causes considerable difficulties in obtaining upper estimates for the critical load coefficient. Below, we construct the functionals Φ and associated velocity fields V_Φ containing a wide class of fields with discontinuities. The functionals Φ on V_Φ will yield the upper bounds for the critical load coefficients. The bounds will be exact in the sense that the lower exact bound of the functional Φ on V_Φ is equal to the critical load coefficient.

The practical value of such functionals is as follows. First, the contraction of Φ on V yields the functional J, hence the upper bound which yields J can be obtained using the functional Φ. Secondly, the functional Φ contains more parameters which can be changed, and this gives greater freedom in constructing the upper bounds for λ_*. Thirdly, computing the functional Φ on the discontinuous fields is not more difficult than computing the functional J on smooth fields. The last factor is very important since the upper bounds obtained on the discontinuous velocity field reduce chiefly to a limiting passage on a sequence of smooth fields converging, in a sense, to the discontinuous velocity field. In fact, such a limiting passage is carried out in the present paper in general terms, and this makes it unnecessary to perform the operation in every particular case. In Sect.2 we give an example in which the use of the above procedure enables us to obtain an exact value of the critical load on the discontinuous field.

Let us describe the general plan for constructing the functionals Φ and sets V_Φ on which they are defined. A major role is played here by the minimax problem with a saddle point, and the value of the corresponding Lagrangian at this saddle point is of the same accuracy as the critical load coefficient. The problem can be called a mathematical extension of the Lagrangian

$$\int_\Omega \varepsilon_{ij}(v)\, \tau_{ij}\, dx$$

and its saddle point can be called the generalized solution of the problem (1.4). However, the set of variations of dual variables constructed in this problem, although containing all possible discontinuities and smooth velocity fields, is not easy to use in practice. It is therefore advisable to separate out certain classes of discontinuities and obtain the intermediate sets of velocity and stress fields containing the smooth fields. Computing the values of the extended Lagrangian on these intermediate sets, we arrive at the functions Φ. Their lower bound on these sets is equal to critical load coefficient.

2. Main results. Let the surface Γ_0 divide the region Ω into two regions Ω^1 and Ω^2, each possessing a boundary satisfying the Lipshits condition. Let us denote by $\nu^1 = (\nu_i^1)$ and $\nu^2 = (\nu_i^2)$ the unit outward normals to the surface bounding the regions Ω^1 and Ω^2, respectively, and by $\nu = (\nu_i)$ the outward unit normal to the surface Γ. Consider a class of functions $V(\Gamma_0, \gamma)$ of the following type:

$$\begin{gathered} V(\Gamma_0, \gamma) = \{(v, w) \colon v = v^1 \in D^2(\Omega^1), \text{ if } x \in \Omega^1 \\ v = v^2 \in D^2(\Omega^2), \text{ if } x \in \Omega^2;\ v_i^1\nu_i^1 + v_i^2\nu_i^2 = 0 \\ \text{on } \Gamma_0,\ v_i\nu_i = 0 \text{ on } \Gamma \setminus \gamma;\ \operatorname{div} v = 0 \text{ in } \Omega; \\ w \in D^2(\Omega);\ w_i\nu_i = v_i\nu_i \text{ on } \gamma,\ w = 0 \text{ on } \Gamma \setminus \gamma; \quad \int_\Omega f_i v_i\, dx + \int_\gamma F_i w_i\, d\Gamma = 1\} \end{gathered} \tag{2.1}$$

We note that the class $V(\Gamma_0, \gamma)$ is non-empty since $(v, v) \in V(\Gamma_0, \gamma)$ for any $v \in V$. The class of functions $V(\Gamma_0, \gamma)$ contains velocity fields which may have slip-type discontinuities on the surfaces Γ_0 and Γ. The subclass of the class $V(\Gamma_0, \gamma)$ for which $v \in D^2(\Omega)$, will be denoted by $V(\gamma)$. Putting

$$S(\nu, v) = \frac{1}{2}(\nu_i v_j + \nu_j v_i)$$

we determine the functionals $\Phi_{\Gamma_0, \gamma}$ and Φ_γ on the classes $V(\Gamma_0, \gamma)$ and $V(\gamma)$, as follows:

$$\Phi_{\Gamma_0, \gamma}(v, w) = \sqrt{2}\, k_* \Big(\int_{\Omega^1} |\varepsilon(v^1)|\, dx + \int_{\Omega^2} |\varepsilon(v^2)|\, dx + \tag{2.2}$$

$$\int\limits_{\Gamma_0} |S(v^1, v^1) + S(v^2, v^2)|\, d\Gamma_0 + \int\limits_{\gamma} |S(v, v - w)|\, d\Gamma + \int\limits_{\Gamma\setminus\gamma} |S(v, v)|\, d\Gamma\Big), \quad \forall (v, w) \in V(\Gamma_0, \gamma)$$

$$\Phi_\gamma(v, w) = \sqrt{2}\, k_* \Big(\int\limits_{\Omega} |\varepsilon(v)|\, dx + \int\limits_{\gamma} |S(v, v - w)|\, d\Gamma + \int\limits_{\Gamma\setminus\gamma} |S(v, v)|\, d\Gamma\Big), \quad \forall (v, w) \in V(\gamma)$$

We have the following assertions (for the proof see Sect.3):

$$\lambda_* = \inf_{V(\Gamma_0, \gamma)} \Phi_{\Gamma_0, \gamma}(v, w) = \inf_{V(\gamma)} \Phi_\gamma(v, w) = \inf_{V} J(v) \tag{2.3}$$

We note that the value of the exact lower bound of the functional $\Phi_{\Gamma_0, \gamma}$ on the set $V(\Gamma_0, \gamma)$ is independent of the choice of Γ_0. Thus when the upper bounds are constructed for λ_* using the functional $\Phi_{\Gamma_0, \gamma}$, we have three parameters v, w and Γ_0 compared with a single parameter v when the functional J is used.

We illustrate the use of the functionals constructed above by considering the following standard plane problem. An annulus is given, the outer boundary of which is clamped and the inner boundary subjected to tangential forces of intensity equal in magnitude to the yield point of the material. In such a problem λ_* is estimated to have a lower bound of unity, with the help of the axisymmetric stress field of special type (see below). Indeed, $\lambda_* = 1$ but, as was shown in /2/, no velocity field belonging to class V exists on which the funcional J would take a value equal to unity. This implies that the proof of the fact that $\lambda_* = 1$ using the functional J and class V only, is of a very tenuous character and based on constructing a special sequence of velocity fields belonging to class V on which the value of the functional J would tend to unity. Using, however the class $V(\gamma)$ and functional Φ_γ, we can show by elementary methods that $\lambda_* = 1$.

We will present the relevant arguments. We introduce the polar ρ, θ -coordinates with the pole at the centre of the annulus, and we have $0 < R_1 \leqslant \rho \leqslant R_2,\ 0 \leqslant \theta \leqslant 2\pi$ for the points of the annulus. We use the following notation:

$$v = (v_\rho, v_\theta), \quad w = (w_\rho, w_\theta), \quad \tau = \begin{Vmatrix} \tau_{\rho\rho} & \tau_{\rho\theta} \\ \tau_{\rho\theta} & \tau_{\theta\theta} \end{Vmatrix}, \quad F = (F_\rho, F_\theta)$$

In the present case γ is the inner boundary of the annulus and $f = 0,\ F = (0, k_*)$. Let us put

$$\tau_* = k_* \Big(\frac{R_1}{\rho}\Big)^2 \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

Then we can establish that $R(\tau_*) = 1$ and hence $\lambda_* \geqslant 1$. Let us take $v_* = 0,\ w_* = (2\pi R_1)^{-1}(0, \varphi(\rho))$ where φ is a function continuously differentiable in $[R_1, R_2]$ and such, that $\varphi(R_1) = 1$ and $\varphi(R_2) = 0$. It is clear that the pair $(v_*, w_*) \in V(\gamma)$ and

$$S(v, w_*) = -(4\pi R_1 k_*)^{-1} \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \text{ on } \gamma$$

Then

$$\Phi_\gamma(v_*, w_*) = \sqrt{2}\, k_* \int\limits_{\gamma} |S(v, w_*)|\, d\Gamma = 1 \geqslant \lambda_*$$

and we have $\lambda_* = 1$.

3. Proof of the results.

We denote by μ_γ the Lebesgue surface measure defined in the usual manner on the surface γ satisfying the Lipshits condition, by Σ_γ the σ-algebra of the subset γ measured with respect to μ_γ, and by $\mathbf{ba}(\gamma, \Sigma_\gamma, \mu_\gamma)$ the Banach space of all finitely additive functions φ defined on Σ_γ which have a bounded variation on γ and perfectly continuous with respect to the measure μ_γ (i.e. if $\gamma_0 \in \Sigma_\gamma$ and $\mu_\gamma(\gamma_0) = 0$, then the variation of φ on γ_0 is also equal to zero). Then we find the integral of the function $g \in L^\infty(\gamma)$ over the finitely additive function $\varphi \in \mathbf{ba}(\gamma, \Sigma_\gamma, \mu_\gamma)$ and the functional

$$g \mapsto \int\limits_{\gamma} g\, d\varphi$$

is a continuous linear functional on $L^\infty(\gamma)$. Moreover, the space conjugated with $L^\infty(\gamma)$ is isometrically isomorphic to the space $\mathbf{ba}(\gamma, \Sigma_\gamma, \mu_\gamma)$ if the total variation of φ on γ is taken as the norm of the latter /4/. We shall denote the norm in Sobolev space $W_2^1(\Omega)^n$ by $|||\cdot|||$. and write

$$V_+ = \{v \in D^2(\Omega):\ v = 0 \text{ on } \Gamma \setminus \gamma\}, \quad V_- = \{v \in W_2^1(\Omega)^n:\ v = 0 \text{ on } \Gamma \setminus \gamma\}$$

Let us suppose that γ is distributed on Γ in such a manner that V_- is dense in V_+ on the norm of the space $D^2(\Omega)$. This condition holds automatically if $\gamma = \varnothing$ or $\gamma = \Gamma$. Next we define two basic spaces shich will be used in the theorem on extension

$$U = \Big\{(u,\ \varphi):\ u = (u_i),\ \varphi = (\varphi_i),\ u_i \in L^{n/(n-1)}(\Omega),$$
$$\varphi_i \in \mathbf{ba}\,(\gamma,\ \Sigma_\gamma,\ \mu_\gamma),\ i = 1, \ldots, n,\ \operatorname{div} u = 0 \text{ in } \Omega,$$
$$\int_\Omega f_i u_i\, dx + \int_\gamma F_i\, d\varphi_i = 1\Big\}$$
$$G = \Big\{(\tau,\ g):\ \tau = (\tau_{ij}),\ g = (g_i),\ \tau \in K,$$
$$g_i \in L^\infty(\gamma),\ \tau_{ij,\,j} \in L^n(\Omega),\ i,\ j = 1, \ldots, n,$$
$$\int_\gamma g_i v_i\, d\Gamma = \int_\Omega (\varepsilon_{ij}(v)\tau_{ij} + v_i \tau_{ij,\,j})\, dx, \quad \forall v \in V_+\Big\}$$

The differential operator $\tau_{ij,\,j}$ appearing in the definition of the space G must be regarded as a distribution. Let us consider the extended Lagrangian

$$L(v, \varphi; \tau, g) = -\int_\Omega \tau_{ij,\,j} v_i\, dx + \int_\gamma g_i\, d\varphi_i$$

on the set $U \times G$.

Theorem. With the assumptions made above, the Lagrangian L has, on the set $U \times G$, a saddle point $(u,\ \Psi) \in U,\ (\sigma,\ h) \in G$ such that

$$L(u,\ \Psi;\ \tau,\ g) \leqslant L(u,\ \Psi;\ \sigma,\ h) \leqslant L(v,\ \varphi;\ \sigma,\ h) \tag{3.1}$$

for any pair $(v,\ \varphi) \in U$ and any pair $(\tau,\ g) \in G$, and

$$\lambda_* = L(u, \Psi; \sigma, h) = \sup_G L(u, \Psi; \tau, g) = \min_U \sup_G L(v, \varphi; \tau, g) \tag{3.2}$$

Proof. We define the following auxilliary sets:

$$K_0 = \{\tau \in K:\ \tau_{kk}(x) = 0 \text{ for nearly all } x \in \Omega\}$$
$$V_m = \{v \in V:\ |||v||| \leqslant m\}$$

By virtue of the standard theorems on saddle points (see e.g. /5/), a saddle point of the Lagrangian

$$l(v, \tau) = \int_\Omega \varepsilon_{ij}(v)\tau_{ij}\, dx$$

exists on the set $V_m \times K_0$ such that

$$l(u_m,\ \tau) \leqslant l(u_m,\ \tau_m) \leqslant l(v,\ \tau_m),\quad u_m \in V_m,\ \tau_m \in K_0 \tag{3.3}$$

for any $v \in V_m,\ \tau \in K_0$. From inequality (1.3) it follows that

$$J(u_m) = \min_{V_m} J(v) = l(u_m, \tau_m) \tag{3.4}$$

This means that the sequence u_m has a uniformly bounded norm in $D^2(\Omega)$. Let us write

$$\varphi_m(\gamma_0) = \int_{\gamma_0} u_m\, d\Gamma, \quad \forall \gamma_0 \in \Sigma_\gamma$$

Then $\varphi_m \in \mathbf{ba}\,(\gamma,\ \Sigma_\gamma,\ \mu_\gamma)^n$ and the norms of φ_m are uniformly in m and $\mathbf{ba}\,(\gamma,\ \Sigma_\gamma,\ \mu_\gamma)^n$. We can therefore choose the stresses such that /4/

$$u_\alpha \underset{A}{\to} u \quad \text{weakly in } L^{n/(n-1)}(\Omega)^n \tag{3.5}$$
$$\varphi_\alpha \underset{A}{\to} \Psi \quad (*) - \text{weakly in } \mathbf{ba}(\gamma, \Sigma_\gamma, \mu_\gamma)^n$$
$$\tau_\alpha \underset{A}{\to} \tau_* \in K_0 \quad (*) - \text{weakly in } L^\infty(\Omega)^{n(n+1)/2}$$

Here A is an ordered set directed so as to increase. If $(\tau,\ g) \in G$, then we have

$$l(u_m,\ \tau) = L(u_m,\ \varphi_m;\ \tau,\ g) \tag{3.6}$$

Passing to the limit in inequality (3.3) and taking into account (3.4)—(3.6), we obtain

$$L(u, \psi; \tau, g) \leqslant \inf_{V_*} J(v) \leqslant l(v, \tau_*) \tag{3.7}$$

for any $(\tau, g) \in G$ and for any function $v \in V_*$ where $V_* = V \cap W_2^1(\Omega)^n$. We also have $(u, \Psi) \in U$ since $\operatorname{div} u = 0$ in Ω and

$$\int_\Omega f_i u_i \, dx + \int_\gamma F_i \, d\Psi_i = 1$$

From the right-hand side of inequality (3.7) it follows that a number λ_0 exists such that

$$l(v, \tau_*) = \lambda_0 \Big(\int_\Omega f_i v_i \, dx + \int_\gamma F_i v_i \, d\Gamma \Big), \quad \forall v \in V_\times \tag{3.8}$$

$$V_\times = \{ v \in V_- : \operatorname{div} v = 0 \operatorname{in} \Omega \}$$

$$\lambda_* \leqslant \inf_{V_*} J(v) \leqslant \lambda_0 \tag{3.9}$$

In particular, the identity (3.8) holds for any solenoidal field from $W_2^1(\Omega)^n$ which vanishes on Γ, and in this case a function $p \in L^2(\Omega)$ exists such that /6/

$$\int_\Omega (p \operatorname{div} v + \varepsilon_{ij}(v) \tau_{*ij}) \, dx = \lambda_0 \int_\Omega f_i v_i \, dx \tag{3.10}$$

for any function $v \in W_2^1(\Omega)^n$ whose trace on Γ is equal to zero. Let us put $\sigma = (p\delta_{ij} + \tau_{*ij})$. Then

$$\sigma \in K, \quad \sigma_{ij,j} = -\lambda_0 f_i \in L^n(\Omega) \tag{3.11}$$

It can be shown that

$$l(v, \sigma) = \lambda_0 \Big(\int_\Omega f_i v_i \, dx + \int_\gamma F_i v_i \, d\Gamma \Big), \quad \forall v \in V_+ \tag{3.12}$$

By virtue of the density conditions imposed, it is sufficient to establish (3.12) for any function $v \in V_-$. Let v be any field belonging to V_-, but such that

$$\int_\gamma \nu_i v_i \, d\Gamma = 0 \tag{3.13}$$

A solenoidal field $u_0 \in W_2^1(\Omega)^n$ exists such that $u_0 = v$. Considering the relation (3.10) for the function $v - u_0$ taking identity (3.8) into account, we find that (3.12) holds for the chosen field v. If v is any field belonging to V_-, then we assume that the function p satisfying (3.10) is defined apart from a constant term. Let us write p in the form

$$p = p_0 + c, \quad \int_\Omega p_0 \, dx = 0$$

and choose c so that (3.12) holds. We choose a function $u_* \in V_-$, such that

$$\int_\Omega \operatorname{div} u_* \, dx = c_1 \neq 0$$

If there is no such function in V_-, then the previous situation can be used to confirm (3.12), since in this case (3.13) holds for any function $v \in V_-$. Let us put

$$c = \frac{\lambda_0}{c_1} \Big(\int_\Omega f_i u_{*i} \, dx + \int_\gamma F_i u_{*i} \, d\Gamma \Big) - \frac{1}{c_1} l(u_*, \sigma_*), \quad \sigma_* = (p_0 \delta_{ij} + \tau_{*ij})$$

Writing v in the form

$$v = v' + \frac{u_*}{c_1} \int_\gamma v_i \nu_i \, d\Gamma$$

we find that the field v' satisfies the condition (3.13) and

$$l(v, \sigma) - \lambda_0 \Big(\int_\Omega f_i v_i \, dx + \int_\gamma F_i v_i \, d\Gamma \Big) = \frac{1}{c_1} \int_\gamma v_i \nu_i \, d\Gamma \Big[l(u_*, \sigma_*) -$$

$$\lambda_0 \Big(\int_\Omega f_i u_{*i} \, dx + \int_\gamma F_i u_{*i} \, d\Gamma \Big) + c \int_\Omega \operatorname{div} u_* \, dx \Big] = 0$$

which completes the proof of (3.12).

From (3.11), (3.12) it follows that $(\sigma, h) \in G$, if $h = \lambda_0 F$. Moreover, $\lambda_0 = \lambda_*$ and finally

$$\lambda_* = L(v, \varphi; \sigma, h), \quad \forall (v, \varphi) \in U \tag{3.14}$$

Relations (3.14) and the left-hand side of the inequality (3.7) now prove the theorem, and relations (2.3) follow from it. Indeed, if $(v, w) \in V(\Gamma_0, \gamma)$, then the pair $(v, \varphi_w) \in U$ where

$$\varphi_w(\gamma_0) = \int_{\gamma_0} w\, d\Gamma, \quad \forall \gamma_0 \in \Sigma$$

Let us obtain an upper estimate of $\sup_G L(v, \varphi_w; \tau, g)$. To show the purpose of this estimate, we assume for simplicity that the region Ω in star-like relative to one of its points. Let $(\tau, g) \in G$. Then a sequence of symmetric tensor fields τ^m exists such that

$$\begin{aligned} &\tau_{kk}^m \to \tau_{kk} \quad \text{strongly in} \quad L^2(\Omega) \\ &\tau_{ij}^{Dm} \to \tau_{ij}^D \ (*)\text{-} \quad \text{weakly in} \quad L^\infty(\gamma) \\ &\tau_{ij,j}^m \to \tau_{ij,j}^m \quad \text{strongly in} \quad L^n(\Omega) \ (i, j = 1, \ldots, n) \\ &\tau_{ij}^m \in C^\infty(\overline{\Omega}), \quad \tau^m \in K \end{aligned} \tag{3.15}$$

To construct such a sequence it is sufficient to carry out the following operations; we place the origin of coordinates at the centre of the star-shaped region Ω, continue the tensor τ by means of a similitude transformation, with the centre at the star, to the region obtained from Ω by applying the same similitude transformation. Then the sequence shown in (3.15) can be constructed using the averages of the contribution of the tensor τ using standard averaging kernels /7/. We further have

$$\begin{aligned} L(v, \varphi_w; \tau^m, g) = &-\int_\Omega v_i \tau_{ij,j}{}^m\, dx + \int_\gamma g_i w_i\, d\Gamma = \int_{\Omega^1} \varepsilon_{ij}(v^1)\tau_{ij}{}^m\, dx + \\ &\int_{\Omega^2} \varepsilon_{ij}(v^2)\tau_{ij}{}^m\, dx - \int_{\Gamma_0} \tau_{ij}{}^m (S_{ij}(v^1, v^1) + S_{ij}(v^2, v^2))\, d\Gamma_0 - \\ &\int_\gamma \tau_{ij}{}^m S_{ij}(\nu, v - w)\, d\Gamma - \int_{\Gamma\setminus\gamma} \tau_{ij}{}^m S_{ij}(\nu, v)\, d\Gamma + \int_\gamma (g_i - \nu_j \tau_{ij}{}^m) w_i\, d\Gamma \end{aligned}$$

By virtue of the definition of the class $V(\Gamma_0, \gamma)$ the first invariants of the tensors $S(v^1, v^1) + S(v^2, v^2)$, $S(\nu, v - w)$ and $S(\nu, v)$ are zero on Γ_0, γ and $\Gamma \setminus \gamma$, respectively. Since $\tau^m \in K$, we arrive at the inequality

$$L(v, \varphi_w; \tau^m, g) \leqslant \Phi_{\Gamma_0, \gamma}(v, w) + \int_\gamma (g_i - \nu_j \tau_{ij}{}^m) w_i\, d\Gamma$$

According to the definition of the class $V(\Gamma_0, \gamma)$ the vector function $w \in V_+$. Therefore by virtue of (3.15) and the definition of the class G, we obtain

$$\int_\gamma (g_i - \nu_j \tau_{ij}{}^m) w_i\, d\Gamma = \int_\Omega (w_i(\tau_{ij,j} - \tau_{ij,j}^m) + \varepsilon_{ij}(w)(\tau_{ij} - \tau_{ij}{}^m))\, dx \to 0$$

But since $L(v, \varphi_w; \tau^m, g) \to L(v, \varphi_w; \tau, g)$, we finally have

$$L(v, \varphi_w; \tau, g) \leqslant \Phi_{\Gamma_0, \gamma}(v, w), \quad \forall (\tau, g) \in G$$

The latter inequality leads to the estimate

$$\sup_G L(v, \varphi_w; \tau, g) \leqslant \Phi_{\Gamma_0, \gamma}(v, w) \tag{3.16}$$

From the estimate (3.16) and statement (3.2) of the theorem it follows that

$$\lambda_* \leqslant \inf_{V(\Gamma_0, \gamma)} \Phi_{\Gamma_0, \gamma}(v, w)$$

and the inverse inequality is obvious. This proves (2.3). We note that more detailed investigations lead to the conclusion that we have an equality in relation (3.16).

In the general case the existence of a sequence possessing the properties shown in (3.15) is proved by standard methods, since the region whose boundary satisfies the Lipshits condition is locally star-like. Remembering that $\overline{\Omega}$ is compact, we can use finite division of unity in $\overline{\Omega}$ to reduce everything to a region that is star-like with respect to one of its points.

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TIME DIFFERENTIATION OF TENSORS DEFINED ON A SURFACE MOVING THROUGH A THREE-DIMENSIONAL EUCLIDEAN SPACE*

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The well-known formulas for the derivatives of Eulerian and Lagrangian basis vectors are used to derive expressions for the derivatives of the surface, volume and double tensors defined on a surface moving through an Euclidean space. In the case of a plane moving through space with constant velocity, the results obtained correspond to the two-dimensional analogs of the results obtained in /1/. A relation connecting the derivatives in question with the derivative $(\delta/\delta t)$ is given, and the concept of the derivative $(\delta/\delta t)$ is introduced for the three-dimensional case.

In /1/ the author developed a theory of the time differentiation of tensors in the three-dimensional case, based on introducing Euclidean and Lagrangian basis vectors and a polyadic representation of tensors in these bases. The problem of the time differentiation of tensors was also considered in /2, 3/ using a general formulation, where a detailed analysis of the earlier work was also given. In /4—6/, in the course of studying wave propagation in continuous media, the derivative $(\delta/\delta t)$ of the components of three-dimensional vectors defined on a surface moving through a three-dimensional Euclidean space (at the wave front) was introduced. The results were generalized in /7/ to the case of surface and dual tensors defined on a moving surface.

1. The law of motion of the points belonging to a three-dimensional continuum is described by the equations

$$x^i = x^i(\xi^1, \xi^2, \xi^3, t), \quad \xi^k = \xi^k(x^1, x^2, x^3, t) \tag{1.1}$$

where x^i are the spatial (Eulerian) coordinates, ξ^k are the material (Lagrangian) coordinates and t is the time. The partial derivatives of the radius vector of the points of the space

$$E_i = \frac{\partial \mathbf{r}}{\partial x^i}, \quad E_i^\wedge = \frac{\partial \mathbf{r}}{\partial \xi^i} \tag{1.2}$$

define, respectively, the fixed Eulerian and moving Lagrangian basis. The tensor $\mathbf{T}$ with a typical distribution of the indices can be represented in invariant form /1/ as

$$\mathbf{T} = T^k_{\cdot m} E_k E^m = T^{\wedge k}_{\cdot m} E_k^\wedge E^{\wedge m} \tag{1.3}$$

The velocity vector of a particle with material coordinates is given by

$$\mathbf{v} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_\xi = v^i E_i = v^{\wedge i} E_i^\wedge; \quad v^i = \left(\frac{\partial x^i}{\partial t}\right)_\xi \tag{1.4}$$

$$v^{\wedge i} = \frac{\partial \xi^i}{\partial x^k} v^k$$

The time derivative of the tensor T can be obtained after establishing the formulas for differentiation of the basis vectors

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